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Variational approximations for gravity waves in water of variable depth

By JOHN MILES
Institute of Geophysics and Planetary Physics, University of California, San Diego, La Jolla, CA 92093-0225, USA

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Eckart's (1952) second-order, self-adjoint partial differential equation for the free-surface displacement of monochromatic gravity waves in water of variable depth h is derived from a variational formulation by approximating the vertical variation of the velocity potential in the average Lagrangian by that for deep-water waves. It is compared with the 'mild-slope equation', which also is second order and self-adjoint and may be obtained by approximating the vertical variation in the average Lagrangian by that for uniform, finite depth. The errors in these approximations vanish for either $\kappa h \downarrow 0$ or $\kappa h \uparrow \infty$ ($\kappa \equiv \omega^2/g$). Both approximations are applied to slowly modulated wavetrains, following Whitham's (1974) formulation for uniform depth. Both conserve wave action; the mild-slope approximation conserves wave energy, but Eckart's approximation does not (except for uniform depth). The two approximations are compared through the calculation of reflection from a gently sloping beach and of edge-wave eigenvalues for a uniform slope (not necessarily small). Eckart's approximation is inferior to the mild-slope approximation for the amplitude in the reflection problem, but it is superior in the edge-wave problem, for which it provides an analytical approximation that is exact for the dominant mode and in error by less than 1.6% for all higher modes within the range of admissible slopes. In contrast, the mild-slope approximation requires numerical integration (Smith & Sprinks 1975) and differs significantly from the exact result for the dominant mode for large slopes.

1. Introduction

Some forty years ago, Carl Eckart (1951, 1952) proposed the approximation

$$\nabla\cdot(H\nabla Z)+K^2HZ=0, \tag{1.1}$$

where

$$H=\frac{1-\mathrm{e}^{-2\kappa h}}{2\kappa},\quad K^2=\kappa^2\coth\kappa h,\quad \kappa=\frac{\omega^2}{g}, \tag{1.2a–c}$$

for the governance of linear gravity waves of free-surface displacement $\zeta = Z(\boldsymbol{x})\sin\omega t$ (an arbitrary phase constant may be added to ωt) in water of variable depth $h(\boldsymbol{x})$. It may be compared with the 'mild-slope equation' (Smith & Sprinks 1975), hereinafter the MSE,

$$\nabla\cdot(\mathscr{H}\nabla Z)+k^2\mathscr{H}Z=0, \tag{1.3}$$

where

$$\mathscr{H}=\tfrac{1}{2}(k^{-1}\tanh kh+h\,\mathrm{sech}^2\,kh),\quad k\tanh kh=\kappa. \tag{1.4a, b}$$

The MSE is exact for uniform depth, but it requires the inversion of (1.4b) at each step in the numerical integration for variable depth. Both (1.1) and (1.3) reduce to Lamb's (1932) shallow-water equation,

$$\nabla\cdot(h\nabla Z)+\kappa Z = 0 \tag{1.5}$$

in the limit $\kappa h \downarrow 0$ or to the Helmholtz equation

$$\nabla^2 Z+\kappa^2 Z = 0 \tag{1.6}$$

in the limit $\kappa h \uparrow \infty$, and both are second order and self-adjoint. In contrast, the partial-differential equation in the exact formulation (*exact* as used herein refers to the exact linear formulation) for variable depth (Miles 1985),

$$\nabla\cdot\left[\left\{\frac{\sinh \not{k} h}{\not{k}}+\kappa\left(\frac{1-\cosh \not{k} h}{\not{k}^2}\right)\right\}\nabla Z\right]+\kappa Z = 0, \tag{1.7}$$

where

$$\not{k}^2 \equiv -(\partial_x^2+\partial_y^2) \tag{1.8}$$

operates only on Z (whereas ∇ operates on both h and Z), is of transcendental order and is not self-adjoint.

Eckart (1952) derived (1.1) by first transforming the exact boundary-value problem to an integral equation for Z and then discarding a presumably small (and intractable) integral 'without examining the justification of this approximation'. He showed that the dispersion relation implied by (1.1) approximates the exact dispersion relation for uniform depth within 4% for $0<\kappa h<\infty$ and that (1.1) provides a description of wave reflection from a beach of uniform slope without the invocation of matched asymptotic expansions (cf. Friedrichs 1948); however, in his 1951 lecture notes he obtained a rather unsatisfactory approximation to the group velocity (see below), and this may have discouraged him from the further development of his approximation. In fact, the direct calculation of $c_g = \partial\omega/\partial k$ from Eckart's dispersion relation $k = K(\omega, h)$ (1.2*b*) yields an approximation to c_g/c_p ($c_p = \omega/k$ is the phase velocity) that is within 1% of the exact result for $0 \leqslant \kappa h < \infty$.

Smith & Sprinks (1975) compare Eckart's equation with the MSE by comparing (in their figure 1) graphs of $\kappa\mathscr{H}$ and $k^2\mathscr{H}/\kappa$ *vs.* κh with those of κH and K^2H/κ and infer from this comparison that the MSE is superior to (1.1) for variable depth. A physically more significant comparison is between

$$KH = \tfrac{1}{2}(1-\mathrm{e}^{-4\kappa h})^{\frac{1}{2}} \tag{1.9}$$

and

$$k\mathscr{H} = \tfrac{1}{2}(\tanh kh+kh\,\mathrm{sech}^2\,kh), \quad \kappa h = kh\tanh kh \tag{1.10a, b}$$

(kh is a parametric variable in (1.10)), to which the wave energy is inversely proportional in the geometrical-optics approximation, and between K/κ and k/κ *vs.* κh. As already noted, K and k differ by less than 4% over $0<\kappa h<\infty$, but KH and $k\mathscr{H}$ differ by as much as 17%, even though both have the correct limiting values, $(\kappa h)^{\frac{1}{2}}$ and $\frac{1}{2}$, respectively, for $\kappa h\downarrow 0$ and $\kappa h \uparrow \infty$. This suggests that Eckart's equation may be satisfactory (and more convenient than the MSE if ω is prescribed) for the prediction of eigenvalues but inferior to the MSE for the prediction of amplitudes.

My aims in the present development are to derive (1.1)–(1.5) from an average-Lagrangian formulation, to elucidate the deficiencies of Eckart's equation in its description of energy propagation, and to compare its predictions in specific examples with those of the MSE. Starting (in §2) from the action integral, which is proportional to the average Lagrangian for harmonic motion, I approximate the vertical variation of the velocity potential by that for deep water. This single approximation (in addition to those implicit in the linearization) yields an average Lagrangian that implies Eckart's equation (1.1). The corresponding approximation of the vertical variation by that for finite, uniform depth yields an average Lagrangian that implies the MSE (1.3), while the neglect of the vertical variation in the Lagrangian implies Lamb's equation (1.5).

It is worth emphasizing that the introduction of a particular approximation in the Lagrangian almost always leads to superior results *vis-à-vis* the introduction of that same approximation in the equations of motion, in part because the error in the Euler–Lagrange equations derived from the variational principle for the Lagrangian is of the order of the square of the error in the trial function, and, even more importantly, because consistent approximations to the symmetries (e.g. conservation of energy) of the exact equations are preserved by their variational counterparts (see Salmon 1988). These advantages become even more significant for nonlinear wave motion.

In §3, I consider a wavetrain for which the amplitude a, the wavenumber k, and the frequency ω are slowly varying (compared with the carrier scales $1/k$ and $1/\omega$) functions of both x and t and h is a slowly varying function of x and, following Whitham (1974), construct the average Lagrangian $\mathscr{L}$ as a function of $a, k \equiv \theta_x$, $\omega \equiv -\theta_t$, and x (owing to the variation of h) for any assumed vertical variation of the velocity potential. The requirements that the action be stationary with respect to independent variations of the amplitude a and the phase θ yields a dispersion relation of the form $D(k, \omega, x) = 0$ and the conservation equation

$$\mathscr{A}_t + (c_g \mathscr{A})_x \equiv \mathscr{C}\mathscr{A} = 0 \tag{1.11}$$

for the wave action $\mathscr{A} \equiv \partial\mathscr{L}/\partial\omega$. Moreover, it follows from Noether's theorem and the invariance of $\mathscr{L}$ under translation of t that the specific energy $\hat{\mathscr{E}} = \omega\mathscr{A}$ is conserved (in the sense that it satisfies $\mathscr{C}\hat{\mathscr{E}} = 0$). This energy proves to be equal to the wave energy $\mathscr{E} = \frac{1}{2}ga^2$ for the MSE, but $\hat{\mathscr{E}} = F(\kappa h)\mathscr{E}$ for Eckart's approximation, where F varies from 1 at $\kappa h = 0$ through a minimum of 0.784 for $\kappa h = 0.575$ to 1 at $\kappa h = \infty$. F may be factored out of the conservation equation for $\hat{\mathscr{E}}$, and $\mathscr{E}$ is conserved, if h is constant, but Eckart's approximation does not conserve $\mathscr{E}$ for variable depth. It is this non-conservation that underlies Eckart's (1951) erroneous approximation to the group velocity, which he obtained by dividing the energy flux (the vertical integral of the mean product of the hydrodynamic pressure and the velocity normal to the wave front) by the mean energy density $\mathscr{E}$; however, it should be emphasized that the role of wave action in this context was realized only with the later work of Whitham (1965), Bretherton & Garrett (1969), and Hayes (1970).

I illustrate Eckart's equation by applying it, in §4, to the reflection problem for a smooth, gently sloping beach of finite offshore depth (Eckart 1952 solves the corresponding problem for a uniform slope) and, in §5, to the edge-wave problem for uniform slope (not necessarily small). As suggested by the preceding comparison of KH and $k\mathscr{H}$, Eckart's equation is inferior to the MSE in its prediction of the amplitude in the reflection problem if the offshore depth is neither shallow nor deep, although both approximations agree with that of Friedrichs (1948) for the limiting case of small, uniform slope (infinite offshore depth). But Eckart's equation is superior to the MSE for the prediction of the edge-wave eigenvalues in that it provides direct, analytical results that are exact for the dominant mode and in error by at most 1.6% for all higher modes over the complete range of admissible slopes, whereas the MSE requires numerical integration and yields a result for the dominant mode that is in error for large slopes (Smith & Sprinks 1975).

2. Variational formulation (monochromatic motion)

We choose the action integral for wave motion of period $T = 2\pi/\omega$ as

$$J = \int_0^T L\mathrm{d}t \equiv T\langle L\rangle, \quad L = \iint \hat{L}\,\mathrm{d}x\,\mathrm{d}y, \tag{2.1a, b}$$

where

$$\hat{L} = -\int_{-h}^{\zeta} [\phi_t + gz + \tfrac{1}{2}(\nabla\phi)^2]\,\mathrm{d}z \tag{2.2}$$

is the Lagrangian density for gravity waves (Luke 1967) after factoring out the fluid density, ϕ is the velocity potential, and ζ is the free-surface displacement. Remarking that ϕ and ζ must be in quadrature, we posit

$$\phi = \Phi(\boldsymbol{x}, z)\cos\omega t, \quad \zeta = Z(\boldsymbol{x})\sin\omega t \tag{2.3a, b}$$

in (2.2), neglect terms of fourth and higher order in Φ and Z, and average over T to obtain

$$\mathscr{L} \equiv \langle \hat{L} - \tfrac{1}{2}gh^2\rangle = \frac{1}{4}\left[2\omega Z(\Phi)_{z=0} - gZ^2 - \int_{-h}^{0}(\nabla\Phi)^2\,\mathrm{d}z\right]. \tag{2.4}$$

Hamilton's principle requires J or, equivalently, $\langle L\rangle$ to be stationary with respect to independent variations of Φ and Z, which implies the exact statement

$$\nabla^2\Phi = 0 \quad (-h < z < 0), \tag{2.5}$$

$$\Phi_z + \nabla h\cdot\nabla\Phi = 0 \quad (z = -h), \tag{2.6}$$

$$\Phi_z = \omega Z, \quad \omega\Phi = gZ \quad (z = 0) \tag{2.7a, b}$$

of the (linearized) gravity-wave problem. The solution of (2.5) and (2.7) is given by (Miles 1985)

$$\Phi = (g/\omega)(\cosh \ell z + \kappa\ell^{-1}\sinh \ell z)\,Z, \tag{2.8}$$

where the operator ℓ^2 is defined by (1.8). Substituting (2.8) into (2.6), we obtain (1.7).

Variational approximations to (1.7) may be obtained by adopting approximations to (2.8) in (2.4) and invoking $\delta\langle L\rangle/\delta Z = 0$. Perhaps the simplest such approximation is $\Phi = (g/\omega)Z$, which is independent of z and leads to Lamb's (1932) shallow-water equation (1.5). The deep-water solution of (2.5) and (2.7) suggests the approximation

$$\Phi(\boldsymbol{x}, z) = (g/\omega)\,Z(\boldsymbol{x})\,\mathrm{e}^{\kappa z}, \tag{2.9}$$

the substitution of which into (2.4) yields

$$\mathscr{L} = \tfrac{1}{4}(g/\omega)^2 H[K^2Z^2 - (\nabla Z)^2], \tag{2.10}$$

where H and K are defined by (1.2). Invoking $\delta\langle L\rangle/\delta Z = 0$, we obtain Eckart's equation (1.1) and the boundary condition

$$H\boldsymbol{n}\cdot\nabla Z = 0 \quad (\boldsymbol{x} \text{ on } \partial S), \tag{2.11}$$

where ∂S is the contact line and $\boldsymbol{n}$ is the normal thereto.

The dispersion relation implied by (1.1) for a straight-crested gravity wave of wavenumber k in water of *uniform* depth is given by $k = K(\omega, h)$ (1.2*b*), which coincides with the exact dispersion relation (1.4*b*) for either $kh \downarrow 0(\kappa \to k^2h)$ or $kh \uparrow \infty$ $(\kappa \to k)$ and differs therefrom by less than 4% for $0 < \kappa h < \infty$ (Eckart 1952, figure 2). It offers the advantage, *vis-à-vis* (1.4*b*), of giving k as an explicit function of ω in those problems for which ω is prescribed.

The ratio of group to phase velocities implied by $k = K(\omega, h)$ for fixed h is

$$\frac{c_g}{c_p} = \frac{k}{\omega}\frac{\partial \omega}{\partial k} = \frac{1}{2}\left(1 - \frac{\kappa h}{\sinh 2\kappa h}\right)^{-1}, \tag{2.12}$$

which differs from the exact result implied by (1.4*b*),

$$\frac{c_g}{c_p} = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right), \tag{2.13}$$

by less than 1% for $0 < \kappa h < \infty$.

The exact solution of (2.5)–(2.7) for uniform depth suggests the approximation

$$\Phi = (g/\omega)\, Z(\boldsymbol{x}) \frac{\cosh k(z+h)}{\cosh kh}, \tag{2.14}$$

in which $k = k(\boldsymbol{x})$ is determined by (1.4*b*) with $h = h(\boldsymbol{x})$ therein. Substituting (2.14) into (2.4) and invoking (1.4*a, b*), we obtain

$$\mathscr{L} = \tfrac{1}{4}(g/\omega)^2 \mathscr{H}[k^2 Z^2 - (\nabla Z)^2], \tag{2.15}$$

which implies the MSE (1.3).

3. Slowly modulated wavetrain

We now assume that h is a slowly varying function of x and, for simplicity, two-dimensional motion and, following Whitham (1974), consider a wavetrain of the form

$$\phi = (g/\omega)\, a f(z; k, \omega, h) \sin \theta, \quad \zeta = a \cos \theta, \tag{3.1\,a, b}$$

where

$$k \equiv \theta_x, \quad \omega \equiv -\theta_t, \tag{3.2\,a, b}$$

and a are (by hypothesis) slowly varying functions of x and t. Substituting (3.1) into (2.2), averaging over θ, and neglecting $O(a^4)$ and the derivatives of a, h, k and ω, we obtain

$$\mathscr{L} = \mathscr{L}(\mathscr{E}, k, \omega, x) = D(k, \omega, x)\, \mathscr{E}, \tag{3.3}$$

where

$$2D = 1 - \kappa^{-1} \int_{-h}^{0} [f'^2(z) + k^2 f^2(z)]\, \mathrm{d}z, \tag{3.4}$$

and

$$\mathscr{E} = \tfrac{1}{2} g a^2 \tag{3.5}$$

is (we anticipate) the corresponding approximation to the mean specific energy. (The direct calculation of $\mathscr{E}$ from (3.1) yields $\mathscr{E} = \frac{1}{2} g a^2 (1 + D)$, which reduces to (3.5) after invoking (3.8).)

The variational principle

$$\delta \iint \mathscr{L}(\mathscr{E}, k, \omega, x)\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{3.6}$$

for the variations $\delta\mathscr{E}$ and $\delta\theta$, with k and ω defined by (3.2), implies the Euler equations

$$\mathscr{L}_{\mathscr{E}} = 0, \quad \partial_t \mathscr{L}_\omega - \partial_x \mathscr{L}_k = 0. \tag{3.7\,a, b}$$

The first of these yields the dispersion relation

$$D(k, \omega, x) = 0. \tag{3.8}$$

The second yields the conservation equation

$$\mathscr{A}_t + (c_g \mathscr{A})_x \equiv \mathscr{C}\mathscr{A} = 0, \tag{3.9}$$

where $$\mathscr{A} \equiv \mathscr{L}_{\omega} = D_{\omega}\mathscr{E} \tag{3.10}$$

is the specific wave action, and

$$c_g = -\frac{D_k}{D_\omega} = \left(\frac{\partial\omega}{\partial k}\right)_x \tag{3.11}$$

is the group velocity. The solution of (3.9)–(3.11) is given by

$$c_g\mathscr{A} = -D_k\mathscr{E} = \kappa^{-1}\mathscr{F}\left(t - \int\frac{\mathrm{d}x}{c_g}\right), \tag{3.12}$$

wherein c_g and D_k ultimately may be expressed as functions of x, the factor κ^{-1} anticipates the form of the subsequent results, and $\mathscr{F}$ is determined by the initial conditions.

The Euler equations (3.7) may be augmented by

$$\partial_t(\omega\mathscr{L}_\omega - \mathscr{L}) - \partial_x(\omega\mathscr{L}_k) = 0, \tag{3.13}$$

which follows from Noether's theorem and the invariance of $\mathscr{L}$ under an arbitrary translation of t. Invoking (3.8), (3.10) and (3.11), we reduce (3.13) to $\mathscr{C}\hat{\mathscr{E}} = 0$, where

$$\hat{\mathscr{E}} \equiv \omega\mathscr{L}_\omega - \mathscr{L} = \omega\mathscr{L}_\omega = \omega\mathscr{A} \tag{3.14}$$

is an energy that may be identified as the Hamiltonian density and reduces to $\mathscr{E}$ if and only if $D_\omega = 1/\omega$.

The vertical-distribution and dispersion functions for the MSE are

$$f = \frac{\cosh k(z+h)}{\cosh kh}, \quad D = \frac{1}{2}\left(1 - \frac{k}{\kappa}\tanh kh\right). \tag{3.15a, b}$$

Substituting (3.15*b*) into (3.12) and invoking (3.8), (3.11) and (3.14), we obtain

$$\mathscr{E} = \hat{\mathscr{E}} = (k\mathscr{H})^{-1}\mathscr{F}\left(t - \int\frac{\mathrm{d}x}{c_g}\right), \tag{3.16}$$

where $k\mathscr{H}$ is given by (1.10*a*).

The counterparts of (3.15*a*, *b*) and (3.16) for Eckart's equation are

$$f = \mathrm{e}^{\kappa z}, \quad D = \tfrac{1}{2}\kappa^{-1}H(K^2 - k^2), \tag{3.17a, b}$$

and $$\mathscr{E} = (KH)^{-1}\mathscr{F}\left(t - \int\frac{\mathrm{d}x}{c_g}\right), \tag{3.18}$$

where KH is given by (1.9). But, instead of $\mathscr{E} = \hat{\mathscr{E}}$, as in (3.16),

$$\hat{\mathscr{E}} = \omega\mathscr{A} = F(\kappa h)\mathscr{E}, \tag{3.19}$$

where $$F(\kappa h) = 1 + \mathrm{e}^{-2\kappa h} - 2\kappa h(\mathrm{e}^{2\kappa h} - 1)^{-1} \tag{3.20}$$

varies from 1 at $\kappa h = 0$ through a minimum of 0.784 at $\kappa = 0.575$ to 1 at $\kappa h = \infty$. It follows that $\mathscr{C}\mathscr{E} \neq 0$, and $\mathscr{E}$ is not conserved, in Eckart's approximation for variable depth.

4. Reflection from a gently sloping beach

We consider oblique reflection from a gently sloping beach for which

$$h(x) \downarrow \sigma x \quad (x/l \downarrow 0), \qquad h(x) \sim h_\infty \quad (x/l \uparrow \infty), \tag{4.1a, b}$$

where $\sigma \ll 1$ is the shoreline ($x = 0$) slope, h_∞ is the offshore depth, and $l = O(h_\infty/\sigma)$ is a characteristic length for the beach ($l = \infty$ and $h = \sigma x$ for a beach of uniform slope). Posing

$$\zeta(x,y,t) = a_0 f(x) \cos(\omega t - k_* y) \quad (0 \leqslant x < \infty, -\infty \, y < \infty) \tag{4.2a}$$

$$\sim A a_0 \cos[(K_\infty^2 - k_*^2)^{\frac{1}{2}} x + \psi] \cos(\omega t - k_* y) \quad (K_\infty x \uparrow \infty), \tag{4.2b}$$

where a_0 is the shoreline amplitude, k_* is the longshore wavenumber, K_∞ is given by (1.2b) with $h = h_\infty$ therein, and A and ψ are to be determined, we reduce (1.1) to

$$(Hf')' + (K^2 - k_*^2) Hf = 0. \tag{4.3}$$

The normalization implicit in (4.2) and the boundary condition (2.11) imply

$$f = 1, \quad Hf' = 0 \quad (x = 0). \tag{4.4a, b}$$

The solution of (4.3) and (4.4) is given by (cf. Miles 1990)

$$f = (K^2 - k_*^2)^{-\frac{1}{4}} \left(\frac{\sigma \chi}{2H}\right)^{\frac{1}{2}} J_0(\chi) + O(\sigma/\kappa l), \quad \chi = \int_0^x (K^2 - k_*^2)^{\frac{1}{2}} \mathrm{d}x. \tag{4.5a, b}$$

Letting $\chi \uparrow \infty$ in (4.5a) and equating the asymptotic approximation to (4.2b), we obtain

$$A = \left(\frac{2\sigma}{\pi}\right)^{\frac{1}{2}} \frac{(\sec \theta_i)^{\frac{1}{2}}}{[1 - \exp(4\kappa h_\infty)]^{\frac{1}{4}}} \tag{4.6a}$$

and
$$\psi = \kappa \int_0^\infty \{[\coth \kappa h - \coth \kappa h_\infty \sin^2 \theta_i]^{\frac{1}{2}} - (\coth \kappa h_\infty)^{\frac{1}{2}} \cos \theta_i\} \, \mathrm{d}x - \tfrac{1}{4}\pi, \tag{4.6b}$$

where $\theta_i = \sin^{-1}(k_*/k_\infty)$ is the angle of incidence. The amplitude A reduces to $(2\sigma/\pi \cos\theta_i)^{\frac{1}{2}}$ for a uniformly sloping beach ($h_\infty = 0$), in agreement with the matched-asymptotic solution of (2.5)–(2.7) (Miles 1990); it differs therefrom by at most 9% for $0 < \kappa h_\infty < \infty$.

The corresponding approximations based on the MSE are obtained by replacing H and K by $\mathscr{H}$ and k in (4.2b)–(4.5), and the results for A and ψ are identical with those of the matched-asymptotic solution of (2.5)–(2.7).

5. Edge waves

The boundary-value problem described by (4.1)–(4.4) also admits a discrete set of eigensolutions for which the reflection condition (4.5) is replaced by the null condition

$$f \to 0 \quad (x \uparrow \infty). \tag{5.1}$$

The eigensolutions of (4.3), (4.4), and (5.1) for $h = \sigma x$ are given by (Eckart 1951)

$$f(x) = \xi^\alpha {}_2F_1(-n, 2\alpha + n + 1; 1; 1 - \xi), \quad \xi = \mathrm{e}^{-2\kappa\sigma x}, \tag{5.2a, b}$$

where ${}_2F_1$ is a hypergeometric (Jacobi) polynomial,

$$\alpha = \frac{1 - 2n(n+1)\sigma^2}{2(2n+1)\sigma^2} \quad (n = 0, 1, \ldots, N), \tag{5.3}$$

$$\frac{\kappa}{\kappa_*} = \frac{(2n+1)\sigma}{[1 + \sigma^2 + 4n^2(n+1)^2\sigma^4]^{\frac{1}{2}}} = \frac{(2n+1)\sin\theta}{[1 + 4n^2(n+1)^2 \sin^2\theta \tan^2\theta]^{\frac{1}{2}}} \quad (\theta \equiv \tan^{-1}\sigma), \tag{5.4}$$

and N is the largest integer for which $\alpha > 0$. We remark that (5.4) reduces to $\kappa/k_* = \sin\theta$ for $n = 0$, in agreement with Stokes (1846), and to $\kappa/k_* = (2n+1)\,\sigma$ in the limit $\sigma \downarrow 0$, in agreement with shallow-water theory (Eckart 1951). The exact result is (Ursell 1952)

$$\kappa/k_* = \sin\,(2n+1)\,\theta \quad (n = 0, 1, \dots, N), \tag{5.5}$$

where N is the largest integer for which $(2n+1)\,\theta < \frac{1}{2}\pi$. The maximum error in (5.4), *vis-à-vis* (5.5), is 1.6% and occurs for $\sigma = \frac{1}{2}$ and $n = N = 1$.

Smith & Sprinks (1975) report numerical approximations to κ/k_*, based on the numerical integration of the MSE, for $n = 0$ and 1. Their result for $n = 1$ is graphically indistinguishable from (5.5) (as also is true for (5.4)), but their result for $n = 0$ departs significantly from (4.5) for $\sigma \geqslant 1$, whereas (5.4) is exact for $n = 0$.

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REFERENCES

Bretherton, F. P. & Garrett, C. J. R. 1969 Wavetrains in inhomogeneous moving media. *Proc. R. Soc. Lond.* A **302**, 529–554.

Eckart, C. 1951 Surface waves on water of variable depth. *Wave Rep. No.* 100. Marine Physical Laboratory of the Scripps Institution of Oceanography, University of California.

Eckart, C. 1952 The propagation of gravity waves from deep to shallow water. In *Gravity Waves: (Proc. NBS Semicentennial Symp. on Gravity Waves, NBS, June 15–18, 1951)*, pp. 165–175. Washington: National Bureau of Standards.

Friedrichs, K. O. 1948 Water waves on a sloping beach. *Commun. Appl. Maths* **1**, 109–134.

Hayes, W. D. 1970 Conservation of action and modal wave motion. *Proc. R. Soc. Lond.* A **320**, 187–208.

Lamb, H. 1932 *Hydrodynamics*, section 193. Cambridge University Press.

Luke, J. C. 1967 A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397.

Miles, J. 1985 Surface waves in basins of variable depth. *J. Fluid Mech.* **152**, 379–389.

Miles, J. 1990 Wave reflection from a gently sloping beach. *J. Fluid Mech.* **214**, 59–66.

Salmon, R. 1988 Hamiltonian fluid mechanics. *Ann. Rev. Fluid Mech.* **20**, 225–256.

Smith, R. & Sprinks, T. 1975 Scattering of surface waves by a conical island. *J. Fluid Mech.* **72**, 373–384.

Stokes, G. G. 1846 Report on recent research in hydrodynamics. In *Mathematical and Physical Papers* (1880), vol. 1, pp. 167–187. Cambridge University Press.

Ursell, F. 1952 Edge waves on a sloping beach. *Proc. R. Soc. Lond.* A **214**, 79–97.

Whitham, G. B. 1965 A general approach to linear and nonlinear dispersive waves using a Lagrangian. *J. Fluid Mech.* **22**, 273–283.

Whitham, G. B. 1974 *Linear and Nonlinear Waves*, section 11.7. Wiley-Interscience.